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# Galerkin's Method for Ordinary Differential Equations Subject to Generalized Nonlinear Boundary Conditions

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## I. INTRODUCTION

In this paper we study nonlinear boundary value problems of the form

$$\dot{x}(t) = f(t, x(t)), \qquad t \in [-1, 1] \tag{1.1}$$

subject to

$$G(x) = 0, \tag{1.2}$$

where $f$ and $G$ are smooth nonlinear mappings, $f\colon [-1, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ and $G\colon X \to \mathbb{R}^n$, where $X = (C[-1, 1], \mathbb{R}^n, \|\cdot\|_\infty)$.

This boundary value problem will be studied via Galerkin's method. More precisely, system (1.1)–(1.2) will be analyzed through finite dimensional approximations of the form

$$(I - Q_k)\begin{pmatrix} \dot{x}_k(\cdot) - f(\cdot, x_k(\cdot)) \\ G(x_k) \end{pmatrix} = 0, \tag{1.3}$$

where $x_k$ belongs to $\pi_k$, a finite dimensional subspace of $(C[-1, 1], \mathbb{R}^n, \|\cdot\|_\infty)$, and $(I - Q_k)$ is a projection mapping $(C[-1, 1], \mathbb{R}^n, \|\cdot\|_\infty) \times \mathbb{R}^n$ onto a subspace of $(C[-1, 1], \mathbb{R}^n, \|\cdot\|_\infty)$ having the same dimension as $\pi_k$.

There is a vast collection of literature concerning the application of Galerkin's method to differential equations [1, 3–12, 16–23]. We make a special reference to the papers of Cesari [4, 5] and Urabe [22, 23]. Our results provide an extension and a unification of previous results of Urabe that were condemned with the determination of periodic solutions to systems of ordinary differential equations [22] and with the solvability of



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ordinary differential equations subject to linear, multipoint boundary conditions [23].

Assuming certain smoothness and regularity conditions on the differential equation as well as on the boundary conditions, we prove that if $\bar{x}$ is an isolated solution of the boundary value problem (1.1)–(1.2), then there exists an index $\bar{k}$ such that for all $k \geqslant \bar{k}$, Eq. (1.3) has a solution $\bar{x}_k$ in $\pi_k$ and that $\{\bar{x}_k\}$ converges uniformly to $\bar{x}$.

A procedure is presented by which the existence of a solution $\bar{x}$ to the boundary value problem (1.1)–(1.2) can be determined based on the solvability of the Galerkin equations. The procedure also provides an error bound between the solution of the boundary value problem and the solution of the Galerkin equations.

The general theory is illustrated using the finite dimensional approximations generated through Legendre expansions.

The results in this paper extend those of Urabe [22, 23] in the following ways:

1. We allow nonlinear boundary conditions.

2. It is proved that the rate at which the Galerkin approximations converge to the solution of the boundary value problem is determined by the smoothness of the function $f$ and on specific properties of the approximating subspaces.

3. The generality of the procedure presented here allows for greater flexibility in the choice of approximating subspaces and it sheds light on the underlying structure of the problem.

## II. Preliminaries

If $Y$ and $Z$ are Banach spaces and $Q: Y \to Z$ is linear, we say it is a projection if $Q^2 = Q$ and $Q$ is continuous. If $Q$ is a projection we write $\ker(Q)$ and $\operatorname{Im}(Q)$ to denote the null space and range of $Q$, respectively. If $F$ is a Fréchet differentiable map we denote its derivative by either $F'$ or $DF$; higher order Fréchet derivatives will be written as $D^{(j)}F$.

The space $(C[-1, 1], \mathbb{R}^n, \|\cdot\|_\infty)$ will be denoted by $X$ and we will write $Y = X \times \mathbb{R}^n$. If $\binom{\phi}{v}$ belongs to $Y$ we write $\|\binom{\phi}{v}\|_\infty = \|\phi\|_\infty + |v|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. We will frequently need to make estimates using the $L^2$ norm. If $\phi$ belongs to $(L^2[-1, 1], \mathbb{R}^n, \|\cdot\|_{L^2})$ and $v$ belongs to $\mathbb{R}^n$ we write $\|\binom{\phi}{v}\|_{L^2} = \|\phi\|_{L^2} + |v|$, where $\|\phi\|_{L^2} = (\int_{-1}^{1} |\phi(t)|^2 \, dt)^{1/2}$. If $\phi: [-1, 1] \to \mathbb{R}^n$ is $m$-times continuously differentiable we write $\|\phi\|_{H^m}$ to denote the Sobolev norm; that is, $\|\phi\|_{H^m} = (\sum_{j=0}^{m} \int_{-1}^{1} |\phi^{(m)}(t)|^2 \, dt)^{1/2}$.

We suppose that $\{P_k\}$ is a sequence of orthogonal projections from $(L^2[-1, 1], \mathbb{R}^n, \|\cdot\|_{L^2})$ into itself such that for each $k \geqslant 1$:

(i) The dimension of the image of $P_k$ is finite dimensional and if $\phi \in \operatorname{Im}(P_k)$ then $\phi$ is infinitely differentiable.

(ii) $P_k$ is continuous from $(C[-1, 1], \mathbb{R}^n)$ into itself with the $\|\cdot\|_\infty$ structure in this space.

(iii) If $x_k \in \operatorname{Im}(P_k)$, then $\dot{x}_k \in \operatorname{Im}(P_{k-1})$.

(iv) There exists a decreasing sequence $\sigma_1(k)$ such that $\lim_{k \to \infty} \sigma_1(k) = 0$ and such that for each continuously differentiable function $\phi: [-1, 1] \to \mathbb{R}^n$ we have that

$$\|(I - P_k)\phi\|_{L^2} \leqslant \sigma_1(k)\|\phi\|_{H^1}.$$

(v) There exists a decreasing sequence $\sigma_2(k)$ such that $\lim_{k \to \infty} \sigma_2(k) = 0$ and such that for every $m$-times continuously differentiable $\phi: [-1, 1] \to \mathbb{R}^n$, with $m \geqslant 3$ we have that

$$\|(I - P_k)\phi\|_{H^1} \leqslant \sigma_2(k)\|\phi\|_{H^m}.$$

Furthermore, for each such $\varphi$ $\lim_{k \to \infty} \|(I - P_k)\varphi\|_{H^2} = 0$.

*Remarks.* (a) It is well known [13] that there exists a contant $c$ such that for every continuously differentiable function $\phi: [-1, 1] \to \mathbb{R}^n$

$$\|\phi\|_\infty \leqslant c\,\|\phi\|_{H^1}.$$

(b) Since $\operatorname{Im}(P_k)$ is finite dimensional we know that for each $k \geqslant 1$ there exists a constant $\beta_k$ such that for each $x_k \in \operatorname{Im}(P_k)$

$$\|x_k\| \leqslant \beta_k\,\|x_k\|_{L^2}.$$

Of course, we also know that there exists a constant $d$ such that for each $x \in (C[-1, 1], \mathbb{R}^n)$

$$\|x\|_{L^2} \leqslant d\,\|x\|_\infty.$$

Definition. For each $k \geqslant 1$, $Q_k: Y \to Y$ is given by

$$Q_k\begin{pmatrix}\phi\\ u\end{pmatrix} = \begin{bmatrix}(I - P_{k-1})\phi\\ 0\end{bmatrix}.$$

We assume that the map $f: [-1, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ in (1.1) is $m$-times continuously differentiable, where $m \geqslant 3$. We assume that $G: X \to \mathbb{R}^n$ is continuously Fréchet differentiable and that for each bounded set $\mathscr{B}$ in $X$ there exists a constant $K(\mathscr{B})$ such that $\| DG(\phi_1) - DG(\phi_2) \|_\infty \leqslant K(\mathscr{B}) \| \phi_1 - \phi_2 \|_\infty$ for each $\phi_1, \phi_2$ in $\mathscr{B}$.

As a matter of notation we introduce the following:

$F: X \to X$ is defined by

$$F(\phi)(t) = f(t, \phi(t)).$$

$\mathscr{F}: X \to Y$ is given by

$$\mathscr{F}(\phi) = \begin{bmatrix} F(\phi) \\ G(\phi) \end{bmatrix}$$

and

$$\mathscr{L}\phi = \begin{bmatrix} \dot{\phi} \\ 0 \end{bmatrix},$$

where the domain of $\mathscr{L}$ consists of the continuously differentiable maps from $[-1, 1]$ into $\mathbb{R}^n$.

It is obvious that with this notation, the boundary value problem (1.1)–(1.2) is equivalent to

$$\mathscr{L}x = \mathscr{F}(x). \tag{2.1}$$

*Remark.* It is known [6] that $\mathscr{F}$ is continuously Fréchet differentiable and that

$$D\mathscr{F}(\phi)\, h = \begin{bmatrix} DF(\phi)\, h \\ DG(\phi)\, h \end{bmatrix},$$

where $DF(\phi)(h)(t) = (\partial f/\partial x)(t, \phi(t))\, h(t)$. It is important to note that this does not hold when the space used is $L^2$ [14].

PROPOSITION 2.1. *Suppose that $\bar{x}$ is a solution of* (1.1)–(1.2) *and that the only solution of the linearized boundary value*

$$\dot{y}(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t))\, y(t), \qquad t \in [-1, 1] \tag{2.2}$$

*subject to*

$$DG(\bar{x})\, y = 0 \tag{2.3}$$

*is the trivial one. Then $[\mathscr{L} - \mathscr{F}'(\bar{x})]$ maps $D(\mathscr{L})$ one-to-one and onto $Y$ and the inverse is continuous.*

*Proof.* It is known [21] that if $\binom{h}{v}$ belongs to $Y$ and $\Gamma: X \to \mathbb{R}^n$ is bounded and linear then the boundary value problem

$$\dot{y}(t = A(t)\, y(t) + h(t), \qquad t \in [-1, 1]$$
$$\Gamma y = v$$

has one and only one solution for each such $\binom{h}{v}$ iff the unique solution of

$$\dot{y}(t) = A(t)\, y(t), \qquad t \in [-1, 1]$$
$$\Gamma y = 0$$

is the trivial one.

We see that the first part of the proposition is an obvious consequences of this fact with $A(t) = (\partial f/\partial x)(t, \bar{x}(t))$ and $\Gamma = DG(\bar{x})$.

Since $\mathscr{L}$ is closed and $D\mathscr{F}(\bar{x})$ is continuous, it follows that $[\mathscr{L} - D\mathscr{F}(\bar{x})]$ is also closed and hence, so is $|\mathscr{L} - D\mathscr{F}(\bar{x})]^{-1}$. Since $Y$ is a Banach space, the continuity of $[\mathscr{L} - D\mathscr{F}(\bar{x})]^{-1}$ follows from the Closed Graph Theorem [15].

PROPOSITION 2.2. *Suppose $\bar{x}$ is a solution of* (1.1)–(1.2) *and that the only solution of* (2.2)–(2.3) *is the trivial one. Then $\bar{x}$ is an isolated solution of* (1.1)–(1.2).

*Proof.* It is clear that if $T$ is defined by

$$T(x) = [\mathscr{L} - \mathscr{F}'(\bar{x})]^{-1}\, [\mathscr{F}(x) - \mathscr{F}'(\bar{x})\, x],$$

then the fixed points of $T$ are precisely the solutions of $\mathscr{L}x = \mathscr{F}(x)$. We observe that $T$ is continuously Fréchet differentiable and that

$$T'(x) = [\mathscr{L} - \mathscr{F}'(\bar{x})]^{-1}\, [\mathscr{F}'(x) - \mathscr{F}'(\bar{x})].$$

Since $\mathscr{F}$ is continuously Fréchet differentiable, it follows that there exists an $r > 0$ such that $\| T'(x) \leqslant \theta < 1$ for all $x \in \{ y : \| y - \bar{x} \| \leqslant r \}$. Also, if $\| x - \bar{x} \| \leqslant r$ then $\| T(x) - \bar{x} \| \leqslant \| T(x) - T(\bar{x}) \| \leqslant \theta \, \| x - \bar{x} \| \leqslant r$.

By the Contraction Mapping Principle we see that $T$ has a unique fixed point in $\{ x \in X \colon \| x - \bar{x} \| \leqslant r \}$. Consequently, $\bar{x}$ is an isolated solution of the boundary value problem (1.1)–(1.2).

For this reason we introduce the following definition.

DEFINITION. A solution $\bar{x}$ of (1.1)–(1.2) is said to be a regular isolated solution if the only solution to the linearized boundary value problem

$$\dot{y}(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t))\, y(t), \qquad t \in [-1, 1]$$

subject to

$$DG(\bar{x})\, y=0$$

is the trivial one.

PROPOSITION 2.3. *Suppose that $\bar{x}$ is a regular isolated solution of the boundary value problem* (1.1)–(1.2). *Then there exists a positive integer $\bar{k}$ such that for all $k \geqslant \bar{k}$ $[\mathscr{L}-\mathscr{F}'(P_k\bar{x})]$ maps $D(\mathscr{L})$ one-to-one and onto $Y$. There exists a constant $M$ such that $\|[\mathscr{L}-\mathscr{F}'(P_k\bar{x})]^{-1}\| \leqslant M$ for all $k \geqslant \bar{k}$ and also $\|[\mathscr{L}-F'(\bar{x})]^{-1}\| \leqslant M$.*

*Proof.* Let $\Phi$ be the principal matrix solution at $t_0=-1$ of the system

$$\dot{y}(t)=\frac{\partial f}{\partial x}(t, \bar{x}(t))\, y(t)$$

and let $\Phi_k$ be the principal matrix solution at $t_0=-1$ of the system

$$\dot{y}(t)=\frac{\partial f}{\partial x}(t, P_k\bar{x}(t))\, y(t).$$

If $\bar{x}$ is a regular isolated solution of (1.1)–(1.2), it follows [21] that the $n$ by $n$ matrix

$$\Psi=(\psi', ..., \psi^n)$$

is nonsingular, where

$$\psi^j=DG(\bar{x})(\phi^j)$$

and $\phi^j$ is the $j$th column of $\Phi$. If $\binom{h}{u}$ belongs to $Y$ we see that the system

$$\dot{y}(t)=\frac{\partial f}{\partial x}(t, \bar{x}(t))\, y(t)+h(t), \qquad t\in[-1, 1]$$

subject to

$$DG(\bar{x})\, y=u$$

has a unique solution and it is given by

$$y(t)=\Phi(t)\, y_0+\Phi(t)\int_{-1}^{t} \Phi^{-1}(s)\, h(s)\, ds,$$

where

$$y_0=\Psi^{-1}u-\Psi^{-1}\left\{DG(\bar{x})\left[\Phi(\cdot)\int_{-1}^{\cdot} \Phi^{-1}(s)\, h(s)\, ds\right]\right\}.$$

This establishes a formula for the map $[\mathscr{L}-\mathscr{F}'(\bar{x})]^{-1}$ whose existence has been established before.

Since we are assuming that $f$ is at least $C^3$ we see that it is a consequence of property (v) and the remark following it that $\lim_{k\to\infty} \|P_k\bar{x}-\bar{x}\|_\infty = 0$. From this it follows (see [9, p. 83]) that $\lim_{k\to\infty} \|\Phi_k - \Phi\|_\infty = 0$. By continuity it follows that there exists a positive integer $\bar{k}$ such that for all $k \geqslant \bar{k}$, $\Psi_k$, defined by

$$\Psi_k = DG(P_k\bar{x})(\Phi_k),$$

is nonsingular. Proceeding as before we see that

$$[\mathscr{L}-\mathscr{F}'(P_k\bar{x})]: D(\mathscr{L}) \to Y$$

is a bijection and its inverse is given by

$$\begin{aligned}
&[\mathscr{L}-\mathscr{F}'(P_k\bar{x})]^{-1}\begin{pmatrix} h \\ u \end{pmatrix}(t) \\
&\quad = \Phi_k(t)\left[\Psi_k^{-1}u - \Psi_k^{-1}\left\{DG(P_k\bar{x})\left[\Phi_k(\cdot)\int_{-1}^{\cdot}\Phi_k^{-1}(s)\,ds\right]\right\}\right] \\
&\qquad + \Phi_k(t)\int_{-1}^{t}\Phi_k^{-1}(s)\,h(s)\,ds.
\end{aligned}$$

Since $\lim_{k\to\infty}\Phi_k = \Phi$ it is clear that there exists a constant $M$ such that $\|[\mathscr{L}-\mathscr{F}'(\bar{x})]^{-1}\| \leqslant M$ and $\|[\mathscr{L}-\mathscr{F}'(P_k\bar{x})]^{-1}\| \leqslant M$ for all $k \geqslant \bar{k}$.

Following the same line of reasoning we arrive at the following result.

PROPOSITION 2.4. *The maps* $[\mathscr{L}-\mathscr{F}'(\bar{x})]^{-1}$ *and* $[\mathscr{L}-\mathscr{F}'(P_k\bar{x})]^{-1}$ *from* $(C[-1,1], \mathbb{R}^n)\times\mathbb{R}^n$ *into* $(C[-1,1], \mathbb{R}^n)$ *with the* $\hat{L}^2$ *and* $L^2$ *structure respectively satisfy*

$$\|[\mathscr{L}-\mathscr{F}'(\bar{x})]^{-1}\| \leqslant M$$

*and*

$$\|[\mathscr{L}-\mathscr{F}'(P_k\bar{x})]^{-1}\| \leqslant M \qquad \text{for all} \quad k \geqslant \bar{k}.$$

Except for trivial details the proof is the same as that of the previous proposition. For this reason it is omitted.

## III. MAIN RESULTS

DEFINITION. For each $k \geqslant 1$ we define

$$S_k: (\operatorname{Im}(P_k), \|\cdot\|_{L^2}) \to (\operatorname{Im}(I-Q_k), \|\cdot\|_{\hat{L}^2})$$

by

$$S_k(x_k) = (I - Q_k)(\mathscr{L} x_k - \mathscr{F}(x_k)).$$

PROPOSITION 3.1. *Suppose $\bar{x}$ is a regular isolated solution of the boundary value problem*

$$\dot{x}(t) = f(t, x(t)), \qquad t \in [-1, 1]$$

*subject to*

$$G(x) = 0.$$

*Then, there exists a positive integer $\bar{k}$ such that for all $k \geqslant \bar{k}$*

$$S'(P_k \bar{x}) : \operatorname{Im}(P_k) \to \operatorname{Im}(I - Q_k)$$

*is a bijection and there exists a constant $\bar{M}$ such that*

$$\|[S'(P_k \bar{x})]^{-1}\| \leqslant \bar{M} \qquad \textit{for all} \quad k \geqslant \bar{k}.$$

*Proof.* Suppose $\binom{y}{v} \in \operatorname{Im}(I - Q_k)$ and that there exists an $x_k \in \operatorname{Im}(P_k)$ such that

$$S'(P_k \bar{x})\, x_k = \begin{pmatrix} y \\ v \end{pmatrix}$$

equivalently,

$$(I - Q_k)(\mathscr{L} x_k - \mathscr{F}'(P_k \bar{x})\, x_k) = \begin{pmatrix} y \\ v \end{pmatrix}.$$

Since $\dot{x}_k \in \operatorname{Im}(P_{k-1})$ it follows that

$$(I - Q_k)(\mathscr{L} x_k) = \begin{pmatrix} P_{k-1} \dot{x}_k \\ 0 \end{pmatrix} = \mathscr{L} x_k.$$

Therefore,

$$\mathscr{L} x_k - \mathscr{F}'(P_k \bar{x})\, x_k = \begin{pmatrix} y \\ v \end{pmatrix} - Q_k(\mathscr{F}'(P_k \bar{x})\, x_k)$$

which implies

$$x_k = [\mathscr{L} - \mathscr{F}'(P_k \bar{x})]^{-1} \left[ \begin{pmatrix} y \\ v \end{pmatrix} - Q_k(\mathscr{F}'(P_k \bar{x})\, x_k) \right].$$

Using Proposition 2.4 we see that for all $k$ large enough

$$\|x_k\|_{L^2} \leqslant M\left[\left\|\begin{pmatrix} y \\ v \end{pmatrix}\right\|_{\hat{L}^2} + \|Q_k(\mathscr{F}'(P_k\bar{x})\, x_k)\|_{\hat{L}^2}\right]$$

$$\leqslant M\left[\left\|\begin{pmatrix} y \\ v \end{pmatrix}\right\|_{\hat{L}^2} + \|(I-P_{k-1})(F'(P_k\bar{x})\, x_k\|_{L^2}\right].$$

We know that

$$\|(I-P_{k-1})(F'(P_k\bar{x})\, x_k)\|_{L^2} \leqslant \sigma_1(k-1)\left\|\frac{d}{dt}(F'(P_k\bar{x})\, x_k)\right\|_{H^1}.$$

Since $f$ is at least three times continuously differentiable, then $\bar{x}$ is four times continuously differentiable and hence (by Property (v), Section II) we have that $\lim_{k\to\infty}\|(I-P_k)\,\bar{x}\|_{H^2}=0$. This together with the fact that there exists a constant $c$ such that $\|\dot{\bar{x}}-(P_k\dot{\bar{x}})\|_\infty \leqslant c\,\|\dot{\bar{x}}-(P_k\dot{\bar{x}})\|_{H_1}$ gives us that $\{\|(d/dt)(P_k\bar{x})\|_\infty\}$ is a bounded sequence.

The above mentioned differentiability of $f$ implies that $F$ is three times continuously differentiable [6] and

$$D^2F(P_k\bar{x})\left(\frac{d}{dt}(P_k\bar{x})\right)(x_k)(t) = D^2f(t, (P_k\bar{x})(t))\left(\frac{d}{dt}(P_k\bar{x})(t)\right)(x_k(t)),$$

where $D^2f(s, y)$ represents the second derivative of $f$ with respect to its second variable. Using this and the fact that $P_k\bar{x}$ converges uniformly to $\bar{x}$ we have that there exists a constant $c_1$ such that

$$\left\|D^{(2)}F(P_k\bar{x})\left(\frac{d}{dt}(P_k\bar{x})\right)(x_k)\right\|_{L^2} \leqslant c_1\,\|x_k\|_{L^2} \qquad \text{for all such } k.$$

It should be observed that this last estimate does not follow from just the smoothness of $F$.

It is clear that

$$\dot{x}_k = F'(P_k\bar{x})(x_k) + y - Q_k(F'(P_k\bar{x})\, x_k.$$

Therefore,

$$\|\dot{x}_k\|_{L^2} \leqslant c_2\,\|x_k\|_{L^2} + \|y\|_{L^2} + \|(I-P_{k-1})(F'(P_k\bar{x})\, x_k)\|_{L^2}$$

for some constant $c_2$.

Consequently,

$$\left\|\frac{d}{dt}(F'(P_k\bar{x})\, x_k)\right\|_{H^1} \leqslant c_1\,\|x_k\|_{L^2} + c_2\,\|x_k\|_{L^2} + \|y\|_{L^2}$$

$$+ \|(I-P_{k-1})(F'(P_k\bar{x})\, x_k)\|_{L^2}$$

and hence

$$\|(I-P_{k-1})(F'(P_k\bar{x})\,x_k)\|_{L^2} \leqslant \sigma_1(k-1)\left[c_1\,\|x_k\|_{L^2}+c_2\,\|x_k\|_{L^2}+\left\|\begin{pmatrix} y \\ v \end{pmatrix}\right\|_{\hat{L}^2}\right]$$
$$+\,\|(I-P_{k-1})(F'(P_k\bar{x})\,x_k)\|_{L^2}\,\sigma_1(k-1).$$

This implies that there exist constants $d_1$, $d_2$ such that

$$\|(I-P_{k-1})(F'(P_k\bar{x})\,x_k)\|_{L^2} \leqslant d_1\sigma_1(k-1)\|x_k\|_{L^2}+d_2\sigma_1(k-1)\left\|\begin{pmatrix} y \\ v \end{pmatrix}\right\|_{\hat{L}^2}$$

and hence

$$\|x_k\|_{L^2} \leqslant M\left[\left\|\begin{pmatrix} y \\ v \end{pmatrix}\right\|_{\hat{L}^2}+d_1\sigma_1(k-1)\|x_k\|_{L^2}+d_2\sigma_1(k-1)\left\|\begin{pmatrix} y \\ v \end{pmatrix}\right\|_{\hat{L}^2}\right].$$

Since $\lim_{k\to\infty}\sigma_1(k)=0$, we have that there exists a constant $\bar{M}$ such that

$$\|x_k\|_{L^2} \leqslant \bar{M}\left\|\begin{bmatrix} y \\ v \end{bmatrix}\right\|_{\hat{L}^2}$$

for all $k$ sufficiently large. This shows that for each such $k$, $S'_k(P_k\bar{x})$ is a bijection and that

$$\|[S'_k(P_k\bar{x})]^{-1}\| \leqslant \bar{M}$$

for all $k \geqslant \bar{k}$.

*Remark.* This result is a generalization of results in [22, 23]. The method of proof is similar to the ones used there. Our proof allows the treatment of nonlinear boundary conditions and imposes less restrictions on the approximating subspaces.

We should recall from Section II that we use $\beta_k$ to denote the constants that satisfy

$$\|x_k\|_\infty \leqslant \beta_k\,\|x_k\|_{\hat{L}^2}$$

for all $x_k \in \mathrm{Im}(P_k)$.

It should also be noted that on $\mathrm{Im}(P_k)$ and on $\mathrm{Im}(I-Q_k)$ we have been using the $L^2$ and $\hat{L}^2$ norms, respectively. Hence, when we write $\|[S'(P_k\bar{x})]^{-1}\|$ we mean the operator norm when the $\mathrm{Im}(P_k)$ and $\mathrm{Im}(I-Q_k)$ are endowed with the $L^2$ and $\hat{L}^2$ structure, respectively.

In the proof of the following theorem we will make estimates using both the $L^\infty$ and $L^2$ norms.

THEOREM 3.2. *Suppose $\bar{x}$ is a reguar isolated solution of* (1.1)–(1.2). *If* $\lim_{k\to\infty} \beta_k \sigma_2(k) = 0$, *then there exists $\bar{k}$ such that for each $k \geqslant \bar{k}$, the Galerkin equations*

$$S_k(x_k) = 0$$

*have solutions $\bar{x}_k$ in* $\mathrm{Im}(P_k)$. *The sequence $\{\bar{x}_k\}$ converges uniformly to $\bar{x}$; more precisely, there exists a constant $C$ such that $\|\bar{x}_k - \bar{x}\|_\infty \leqslant C\sigma_2(k)$.*

*Proof.* We have seen that there exists a constant $M$ such that

$$\|[S'(P_k \bar{x})]^{-1}\| \leqslant M$$

for all $k$ sufficiently large. From the assumptions on $f$ and $G$ it follows that for each $r > 0$, there exists a constant $K(r)$ such that if $\|x\|_\infty$, $\|y\|_\infty \leqslant r$ then $\|\mathscr{F}'(x) - \mathscr{F}'(y)\| \leqslant K(r)\|x - y\|_\infty$. Since $\|P_k \bar{x} - \bar{x}\|_\infty$ converges to zero as $k \to \infty$, it follows that if $\{\delta_k\}$ is a sequence of positive numbers such that $\overline{\lim}_{k\to\infty} (\delta_k \beta_k) < \infty$ then there exists a number $\bar{r} > 0$ such that if $\{y_k\}$ is any sequence such that $y_k \in \mathrm{Im}(P_k)$ and $\|y_k - P_k \bar{x}\|_{L^2} \leqslant \delta_k$ we have that $\sup_k \|y_k\|_\infty \leqslant \bar{r}$.

Let $\{\delta_k\}$ be such a sequence and define

$$T_k \colon \{x \in \mathrm{Im}(P_k) | \, \|x - P_k \bar{x}\|_{L^2} \leqslant \delta_k\} \to \mathrm{Im}(P_k)$$

by

$$T_k(x) = x - [S'_k(P_k \bar{x})]^{-1} S_k(x).$$

It follows that

$$\begin{aligned} T'_k(x) &= I - [S'_k(P_k \bar{x})]^{-1} S'_k(x) \\ &= [S'_k(P_k(\bar{x})]^{-1} [S'_k(P_k(\bar{x}) - S'_k(x)] \\ &= [S'_k(P_k \bar{x})]^{-1} [(I - Q_k)(\mathscr{F}'(x) - \mathscr{F}'(P_k \bar{x}))]. \end{aligned}$$

It is clear that there exists a positive constant $b$ such that if $\delta_k \leqslant b/\beta_k$ then for all $k \geqslant \bar{k}$ we have that

$$\sup_{k \geqslant \bar{k}} \sup_{\{y_k \in \mathrm{Im}(P_k) \colon \|y_k - P_k \bar{x}\|_{L^2} \leqslant \delta_k\}} \|T'_k(y_k)\| \leqslant \tfrac{1}{2}.$$

If $x \in \mathrm{Im}(P_k)$ and $\|x - P_k \bar{x}\|_{L^2} \leqslant \delta_k$ we have that

$$\begin{aligned} \|T_k(x) - P_k \bar{x}\|_{L^2} &\leqslant \|T_k(x) - T_k(P_k \bar{x})\|_{L^2} + \|T_k(P_k \bar{x}) - P_k \bar{x}\|_{L^2} \\ &\leqslant \frac{\delta_k}{2} + M \, \|S_k(P_k \bar{x})\|_{L^2}. \end{aligned}$$

We see that if $\| S_k(P_k\bar{x})\|_{L^2} \leq \delta_k/2M$ then $T_k$ maps

$$\{x \in \operatorname{Im}(P_k) \colon \| x - P_k\bar{x}\|_{L^2} \leq \delta_k\}$$

into itself and it is a contraction. It is obvious that if $T_k$ is a contraction on this set, the unique fixed point of $T_k$ will be a solution of the Galerkin equations. We now proceed to show that there exists a sequence $\{\delta_k\}$ such that $T_k$ is a contraction on $\{x \in \operatorname{Im}(P_k)\colon \|x - P_k\bar{x}\|_{L^2} \leq \delta_k\}$. Furthermore, it will be shown that the sequence $\delta_k$ is such that $\lim_{k\to\infty} \delta_k\beta_k = 0$. Since $\|x - P_k\bar{x}\|_\infty \leq \beta_k \|x - P_k\bar{x}\|_{L^2}$, this implies that the solution of the Galerkin equations, $\bar{x}_k$, satisfy $\lim_{k\to\infty} \|\bar{x}_k - P_k\bar{x}\|_\infty = 0$.

We have seen that

$$\begin{aligned}
\| S_k(P_k\bar{x})\|_{L^2} &= \|(I - Q_k)(LP_k\bar{x} - \mathscr{F}(P_k\bar{x})\|_{L^2} \\
&= \| LP_k\bar{x} - (I - Q_k)\,\mathscr{F}(P_k\bar{x})\|_{L^2} \\
&\leq \| LP_k\bar{x} - L\bar{x}\|_{L^2} + \|\mathscr{F}(\bar{x}) - \mathscr{F}(P_k\bar{x})\|_{L^2} + \|Q_k\mathscr{F}(P_k\bar{x})\|_{L^2} \\
&\leq \|(I - P_k)\,\bar{x}\|_{L^2} + b\,\|(I - P_k)\,\bar{x}\|_\infty + \|(I - P_{k-1})\,F(P_k\bar{x})\|_{L^2}
\end{aligned}$$

for some constant $b$. We know that if $f$ is $C^m$ then $\bar{x}$ is $C^{m+1}$. Therefore, it follows from property (v) and the remark following it, that $\| S_k(P_k\bar{x})\|_{L^2}$ is $\mathcal{O}(\sigma_2(k))$. If $\beta_k\sigma_2(k) \to 0$ as $k \to \infty$ it is clear that a sequence $\delta_k$ can be chosen so that $\| S_k(P_k\bar{x})\|_{L^2} \leq \delta_k/2M$ and $\lim_{k\to\infty} \delta_k\beta_k = 0$. From this it follows that the solution of $S_k(x_k) = 0$, $\bar{x}_k$, satisfies $\|\bar{x}_k - \bar{x}\|_\infty = \mathcal{O}(\sigma_2(k))$.

In the next theorem conditions are established for the existence of a solution to the boundary value problem (1.1)–(1.2) based on the existence of a solution to the Galerkin equations.

THEOREM 3.3. *Suppose $G$ is twice continuously Fréchet differentiable and that $\bar{y}_k \in \operatorname{Im}(P_k)$ is a solution of the Galerkin equations, $S_k(x_k) = 0$, that $[\mathscr{L} - \mathscr{F}'(\bar{y}_k)]^{-1}$ exists and that there exists a number $\delta > 0$ such that*:

(i) $\|[\mathscr{L} - \mathscr{F}'(\bar{y}_k)]^{-1}\| \, \|Q_k\mathscr{F}(\bar{y}_k)\| \leq \delta/2$ *and*

(ii) $\|[\mathscr{L} - \mathscr{F}'(\bar{y}_k)]^{-1}\| \sup_{\{x \in X \colon \|x - \bar{y}_k\| \leq \delta\}} \|D^{(2)}\mathscr{F}(x)\|\, \delta < 1$.

*Then, there exists $\bar{z} \in \{x \colon \|x - \bar{y}_k\|_\infty \leq \delta\}$ which solves the boundary value problem* (1.1)–(1.2). *The solution $\bar{z}$ can be obtained as the uniform limit of the sequence $\{z_n\}$, where $z_0 = \bar{y}_k$ and $z_{n+1} = [\mathscr{L} - \mathscr{F}'(\bar{y}_k)]^{-1} [\mathscr{F}(z_n) - \mathscr{F}'(\bar{y}_k)\, z_n]$.*

*Proof.* For each $x \in \{x \in X\colon \|x - \bar{y}_k\| \leq \delta\}$ define $T(x) = [\mathscr{L} - \mathscr{F}'(\bar{y}_k)]^{-1}\, [\mathscr{F}(x) - \mathscr{F}'(\bar{y}_k)\, x]$. It is straightforward to verify that $\sup_{\{x \in X \colon \|x - \bar{y}_k\| \leq \delta\}} \| T'(x)\| < 1$.

We see that

$$\begin{aligned}T(x)-\bar{y}_k &= [\mathscr{L}-\mathscr{F}'(\bar{y}_k)]^{-1}\,[\mathscr{F}(x)-\mathscr{F}'(\bar{y}_k)\,x-(\mathscr{L}-\mathscr{F}'(\bar{y}_k))\,\bar{y}_k]\\ &= [\mathscr{L}-\mathscr{F}'(\bar{y}_k)]^{-1}\,[\mathscr{F}(x)-\mathscr{F}(\bar{y}_k)-\mathscr{F}'(\bar{y}_k)(x-\bar{y}_k)\\ &\quad +(I-P_{k-1})\,F(\bar{y}_k)]\end{aligned}$$

because

$$\mathscr{L}\bar{y}_k=(I-Q_k)\,\mathscr{F}(\bar{y}_k)$$

and

$$\|Q_k\mathscr{F}(\bar{y}_k)\|=\|(I-P_{k-1})\,F(\bar{y}_k)\|.$$

Therefore,

$$\begin{aligned}\|T(x)-\bar{y}_k\| \leqslant \|[\mathscr{L}-\mathscr{F}'(\bar{y}_k)]^{-1}\|\,[\|(I-P_{k-1})\,F(\bar{y}_k)\|\\ +\sup_{\{x\in X:\,\|x-\bar{y}_k\|\leqslant\delta\}}\|D^2\mathscr{F}(x)\|\,\|x-\bar{y}_k\|^2\,(\tfrac{1}{2})]\leqslant\delta.\end{aligned}$$

From this we see that $T$ maps $\{x\in X\colon \|x-\bar{y}_k\|\leqslant\delta\}$ into itself and that it is a contraction. It is trivial to verify that the fixed point of $T$ is a solution of the boundary value problem (1.1)–(1.2). The remaining part of the theorem is a trivial consequence of the Contraction Mapping Principle.

It is easy to verify that if $\bar{x}$ is a regular isolated solution of the boundary value problem (1.1)–(1.2) the conditions of the last theorem will be satisfied for some index $k$.

We will now present a sequence of projections that satisfy conditions (i)–(v) in Section II.

The Legendre polynomials [3] can be defined on the interval $[-1,1]$ by the formulas

$$L_0(t)=1,\qquad L_1(t)=t$$

for all $t\in[-1,1]$ and

$$L_{k+1}(t)=\left(\frac{2k+1}{k+1}\right)tL_k(t)-\left(\frac{k}{k+1}\right)L_{k-1}(t).$$

It is known that $|L_k(t)|\leqslant 1$ for every $t\in[-1,1]$ and that $L_k(\pm 1)=(\pm 1)^k$.

For each $\phi\in X$ define

$$(P_k\phi)(t)=\sum_{j=0}^{k}L_j(t)\left(j+\frac{1}{2}\right)\int_{-1}^{1}\phi(s)\,L_j(s)\,ds.$$

In [3] the following results are found:

(a) If $\phi$ is $m$-times continuously differentiable, then, for $m \geqslant 1$

$$\|(I-P_k)\,\phi\|_{L^2} \leqslant Ck^{-m}\,\|\phi\|_{H^m}.$$

(b) If $\phi$ is $m$-times continuously differentiable,

$$\|(I-P_k)\,\phi\|_{H^l} \leqslant Ck^{2l-m-1/2}\,\|\phi\|_{H^m}.$$

In both (a) and (b), the constant $C$ is independent of $\phi$.

It is clear that conditions (i)–(iii) in Section II are satisfied when we use $P_k\phi$ as the $k$th-order truncated Legendre expansion of $\phi$. If $\phi$ is continuously differentiable, we see that condition (iv) is satisfied if we take $\sigma_1(k) = Ck^{-1}$; of course, if $\phi$ is smoother the rate of convergence is better. If $\phi$ is $m$-times continuously differentiable with $m \geqslant 3$ we can take $\sigma_2(k) = \mathbb{C}k^{(3/2)-m}$.

To estimate the constants $\beta_k$ in Section II we note that if $\phi \in \operatorname{Im}(P_k)$ then $\phi(t) = \sum_{j=0}^{k} \alpha_j L_j(t)$, where $\alpha_j \in \mathbb{R}^n$ for each $j = 0, 1, \ldots, k$.

$$|\phi(t)| \leqslant \sum_{j=0}^{k} |\alpha_j|\,|L_j(t)| \leqslant \sum_{j=0}^{k} |\alpha_j|.$$

Since $\{L_j\}$ are orthogonal we have that $\|\phi\|_{L^2} = (\sum_{j=0}^{k} |\alpha_j|^2)$. Since

$$\sum_{j=0}^{k} |\alpha_j| \leqslant \left(\sum_{j=0}^{k} |\alpha_j|^2\right)^{1/2} (k+1)^{1/2}$$

we obtain that $\beta_k \leqslant (k+1)^{1/2}$. From Theorem 3.2 we obtain that if $\lim_{k\to\infty} (k+1)^{1/2}\,k^{(3/2)-m} = 0$, then the Galerkin equations have solutions, these solutions converge uniformly to $\bar{x}$, provided the latter is a regular isolated solution, and the error is of the order of $k^{(3/2)-m}$. Certainly if $f$ is three times continuously differentiable the above conditions will be satisfied.

It seems worthwhile to investigate the existence and behavior of solutions to the Galerkin equations when projections associated with eigenfunction expansions for generalized boundary conditions [2, 18, 19] are used.

## References

1. S. Bancroft, J. K. Hale, and D. Sweet, Alternative problems for nonlinear functional equations, *J. Differential Equations* **4** (1988), 40–56.
2. R. N. Bryan, A nonhomogeneous linear differential system with interface conditions, *Proc. Amer. Math. Soc.* **22** (1969), 270–276.

3. C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zany, "Spectral Methods in Fluid Dynamics," Springer-Verlag, New York, 1988.
4. L. Cesari, Functional analysis and periodic solutions of nonlinear differential equations, *Contr. Differential Equations* **1** (1963), 149–187.
5. L. Cesari, Functional analysis and Galerkin's method, *Michigan Math. J.* **11** (1964), 385–414.
6. T. M. Flett, "Differential Analysis," Cambridge Univ. Press, Cambridge, 1980.
7. K. Gustafson and D. Sather, Large nonlinearities and monotonicity, *Arch. Rational Mech. Anal.* **48** (1972), 109–122.
8. K. Gustafson, Large nonlinearities and closedness, *Arch. Rational Mech. Anal.* **52** (1973), 10–19.
9. J. K. Hale, "Ordinary Differential Equations," Wiley–Interscience, New York, 1969.
10. J. K. Hale, Applications of alternative problems, *in* "Lecture Notes," Vol. 71-1, Brown University, Providence, RI, 1971.
11. P. Linz, "Theoretical Numerical Analysis," Wiley–Interscience, New York, 1979.
12. J. Locker, An existence analysis for nonlinear equations in Hilbert space, *Trans. Amer. Math. Soc.* **128** (1967), 403–413.
13. J. Locker, "Functional Analysis and Two-Point Differential Operators," Wiley, New York, 1986.
14. R. H. Martin, Jr., "Nonlinear Operators and Differential Equations in Banach Spaces," Wiley–Interscience, New York, 1976.
15. L. Niremberg, Functional analysis, *in* "Lecture Notes," New York Univ., New York, 1960–1961.
16. J. Osborn and D. Sather, Alternative problems for nonlinear equations, *J. Differential Equations* **17** (1975), 12–31.
17. J. Rodriguez, An alternative method for boundary value problems with large nonlinearities, *J. Differential Equations* **43**, No. 2 (1982), 157–167.
18. J. Rodriguez, Nonlinear differential equations under Stieltjes boundary conditions, *Nonlinear Anal.* **7**, No. 1 (1983), 107–116.
19. J. Rodriguez and D. Sweet, Projection methods for nonlinear boundary value problems, *J. Differential Equations* **58**, No. 2 (1985), 282–293.
20. D. Sweet, "Choosing Projection Operators for the Alternative Method" (L. Cesari, R. Kannan, and J. Schuur, Eds.), Dekker, New York/Basel, 1976.
21. D. Sweet, An alternative method for weakly nonlinear boundary value problems, *Nonlinear Anal.* **8**, No. 5 (1984), 421–428.
22. M. Urabe, Galerkin's procedure for nonlinear periodic systems, *Arch, Rational Mech. Anal.* **20** (1965), 120–152.
23. M. Urabe, Numerical solution of multipoint boundary value problems in Chebyshev series, Theory of the method, *Numer. Math.* **9** (1967), 341–366.